On the theory of growing cavities behind hydrofoils

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A general theory of unsteady cavitating flow past hydrofoils and other obstacles is given for the case of cavities of finite length L. If the circulation Γ, the cavity volume V and L are known as functions of time, the theory yields explicit formulae for the velocity over the wetted surface and for the cavitation number σ. The theory is based on the approximation that the cavity is bounded by streamlines, and so is valid only for slow rates of change of L, V and Γ. The possibility of allowing for the presence of a vortex sheet behind the cavity is discussed. The theory is extended to the case of a cascade of hydrofoils behind which extend growing cavities.

Two examples of the theory are discussed, namely the unsteady flow past a symmetrical wedge, and the unsteady flow past a flat plate hydrofoil cavitating from the leading edge.

1. Introduction

Unsteady cavitating flow is perhaps the least developed branch of incompressible fluid dynamics, and even in two-dimensional problems the number of references, at least in British and American journals, is not large. The main difficulty of the subject lies in the rather involved boundary conditions, and in the fact that the free boundaries enclosing the cavity are material lines and not streamlines. A further complicating feature is the fact that growing cavities require a sink at infinity to accommodate the displaced liquid and this in turn leads to logarithmically infinite pressures at infinity, a point we shall return to shortly. Unfortunately the author is not familiar with the Russian contributions to this field, although the excellent Russian progress in hydrofoil development suggests that unsteady problems have received considerable attention. The references cited below, therefore, are merely those the author has found useful in developing his own work.

Unsteady cavitating flows, in which the cavity extends to infinity, have been studied by Woods (1955), and applied to a problem involving an oscillating stalled aerofoil and to the impulsive motion of a flat plate (see Woods 1961, pp. 454–77). Curle (1956*a*, *b*) has also made contributions to this subject. On finite cavities one can cite Gilbarg's (1952) paper extending some earlier work of von Kármán on symmetrical cusped cavities (negative cavitation number σ). Gilbarg pointed out that it is physically reasonable to assume that the errors introduced by replacing the material lines by streamlines are negligible for slowly varying flows. Woods (1953) extended Gilbarg's work to apply to Riabouchinsky flows (cavity between two bodies), for which σ has positive and therefore physically realistic values. However, both Woods and Gilbarg neglected the sink at infinity, necessarily present if cavity volumes are changing, and in a private communication Dr M. J. Lighthill questioned the validity of a theory which neglected an obviously important feature of unsteady flow.

This defect is removed in this paper and the author's earlier work on finite cavities is extended to apply to asymmetric flows, so that the unsteady flow past a hydrofoil of general shape can be considered. The theory is applied to the case of accelerating symmetric flow past a two-dimensional wedge, and for the particular case of constant cavity volume, the results obtained agree with those due to Cumberbatch (1961) obtained by a different method.

The main gap in the present theory is lack of knowledge of how Q, the rate of change of cavity volume, is linked to such phenomena as the vaporization rate on the cavity walls, and the rate at which the vapour and gases in the cavity are removed by entrainment at the rear end of the cavity. However, as Dr T. Brooke Benjamin has pointed out to the author (see following paper, Brooke Benjamin 1964), in a real flow Q will necessarily depend on factors outside the scope of a two-dimensional theory. There is bound to be an inflow or outflow in the (finite) spanwise direction of the quasi two-dimensional flow past a hydrofoil, and Q will depend on this three-dimensional effect. And further in three dimensions, the logical difficulty of the need to have infinite pressures at infinity in order to produce a growth in the cavity volume will not arise. In his paper Brooke Benjamin shows how one can match a three-dimensional solution for the 'outfield' with the two-dimensional 'infield' solution. Thus in the present paper, which deals with the latter problem, $Q(t)$ must be regarded as an *arbitrary* property of the idealized two-dimensional flow, and the solution given is then a valid representation only of the near flow field.

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2. The boundary-value problem

In the z-plane of figure 1 is shown the hydrofoil, with a wetted surface S_2AS_1 and a cavity $S_1CBB'C'S_2$. Over the surface $CBB'C'$ rapid changes occur in the velocity and direction of the flow, corresponding to the usual turbulent mixing and entrainment region at the end of the cavity. This region will be assumed small compared with the rest of the cavity, and we shall be content to prescribe the velocity over its surface so as to secure only one essential feature of a real cavitating flow, namely the closure of the bubble. The free surfaces S_1C and S_2C' are at constant pressure at a given instant of time, and in unsteady flow they are composed of material lines rather than streamlines; however, as we are mainly interested in the downstream development of the cavity, little error will be introduced by assuming that the free surfaces are stream surfaces. The error due to this approximation will be largest at the rear of the cavity, where the motion of the surface is at a large angle to the fluid motion.

With this approximation then, we can take the curves AS_1CB and $AS_2C'B'$ to lie on the streamline $\psi = 0$ and so arrive at the w-plane (where w is the complex stream function) shown in the figure. Notice that because of circulation the points B and B' do not correspond in this plane. The w-plane is mapped into the semi-infinite strip $-\pi < \gamma \leqslant \pi$, $0 \leqslant \eta < \infty$ ($\zeta = \gamma + i\eta$) of the ζ-plane shown in figure 1 by

$$w = 2a\{\cos\alpha - \cos(\zeta+\alpha) + (\zeta + 2\alpha)\sin\alpha\}, \tag{1}$$

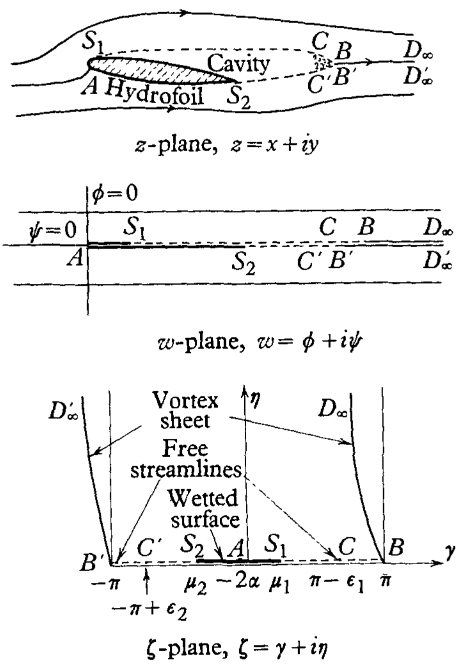

FIGURE 1. The independent variables.

where $w = 0$, $\gamma = -2\alpha$, defines the position of the front stagnation point A, and the (clockwise) circulation is

$$\Gamma = 4\pi a \sin\alpha. \tag{2}$$

For small values of Γ the potential difference between the front stagnation point and the rear of the cavity is approximately $4a$, i.e. the over-all length of the hydrofoil and cavity is

$$L \simeq 4a/V, \tag{3}$$

where V is an average velocity along the streamline AS_1CB. The surfaces of the hydrofoil and cavity are mapped on to $\eta = 0$, while the trailing edge streamline CD_∞ is mapped on to the curve $\eta\sin\alpha + \sinh\eta\sin(\gamma+\alpha) = 0$. When the circulation, and hence α, is small, these curves lie close to $\gamma = \pm\pi$, so that boundary conditions on CD_∞ can be applied on $\gamma = \pm\pi$ with little error (cf. the boundary approximation of linear perturbation theory for aerofoils).

Let (q, θ) be the velocity vector in polar co-ordinates, with θ measured from the flow direction at infinity, and let

$$\tau = \ln(U\,dz/dw) = \Omega + i\theta, \quad \Omega \equiv \ln(U/q), \tag{4}$$

where U is the velocity at infinity. Then τ is an analytic function of ζ that satisfies the following boundary conditions:

$$
\begin{array}{rl}
\text{(i)} & \Omega(\gamma) = \Omega_s(\gamma) \quad \text{in} \quad -\pi < \gamma < \mu_2, \quad \mu_1 < \gamma < \pi, \quad \eta = 0,\\
\text{(ii)} & \theta(\gamma) = \theta_s(\gamma) \quad \text{in} \quad \mu_2 < \gamma < \mu_1, \quad \eta = 0,\\
\text{(iii)} & \lim\limits_{\eta \to \infty} \tau(\gamma + i\eta) = 0,\\
\text{(iv)} & \lim\limits_{\gamma \to \pi} \tau(\gamma + i0) < \infty,\\
\text{(v)} & d\Gamma/dt = 0 \text{ (Kelvin's theorem)},\\
\text{(vi)} & \theta(\pi + i\eta) - \theta(-\pi + i\eta) = 0, \quad 0 < \eta < \infty,\\
\text{(vii)} & \Omega(\pi + i\eta) - \Omega(-\pi + i\eta) = \omega(\eta), \quad 0 < \eta < \infty,\\
\text{(viii)} & \displaystyle\int_{-\pi+i\eta}^{\pi+i\eta} dz = 0,\\
\text{(ix)} & \displaystyle\int_{-\pi+i\eta_1}^{\pi+i\eta_1} (dw/d\zeta)\, d\zeta = \Gamma - iQ.
\end{array}
$$

Here Q is the rate at which the volume per unit breadth of the cavity is increasing, i.e. Q is the strength of the sink at infinity.

In (i), $\theta_s(\gamma)$ is assumed to be a known function of γ, but with curved wetted surfaces this function will not be known exactly (see remarks in §4). To determine $\Omega_s(\gamma)$ to use in (ii) we shall use Bernoulli's equation, and the fact that the cavity pressure depends only on the time. Condition (iii) is obvious; (iv) ensures that there is no stagnation point at the rear of the cavity, i.e. that the cavity is locally cusped. In (v) the circulation is taken about any contour $\mathscr{C}$, completely enclosing the hydrofoil and cavity, and moving with the fluid. Recall that in unsteady *aerofoil* theory, the presence of a vortex sheet springing from the (solid) trailing edge B (see figure 1) and lying along the streamline BD_∞, prevents one deforming $\mathscr{C}$ so as to alter the fluid element R, say, at which $\mathscr{C}$ intersects BD_∞. As R is convected downstream, $\mathscr{C}$ is continuously enlarged. In general it is also necessary in the present hydrofoil problem to postulate the existence of a vortex sheet along BD_∞; otherwise it would not be possible to change the steady lift force

$$\mathscr{L} = \rho U \Gamma = 4aU\pi\rho \sin\alpha, \tag{5}$$

on the hydrofoil to another (steady) value. For this reason we have made allowance in (vii) for a discontinuity in q—and hence in Ω—across BD_∞, or rather across $\gamma = \pm\pi$ (see remarks following (3)). Note that corresponding to (5) there is a steady drag force D acting on the hydrofoil and wake together given by

$$D = -\rho U Q. \tag{6}$$

In (viii) we have the obvious closure condition for the hydrofoil-cavity 'body'. Finally (ix) expresses the discontinuity in $w = \phi + i\psi$ about a circuit like $\mathscr{C}$, the point $\zeta = \eta_1$ having to move with the fluid if Γ on the right-hand side is to be independent of the time t. Notice that if (ix) is to be satisfied it is necessary to assume that $dw/d\zeta$ is of the form

$$dw/d\zeta = iae^{-i(\zeta+\alpha)} + (2\pi)^{-1}(\Gamma - iQ) - iae^{i(\zeta+\alpha)} + O(Qe^{2i\zeta}), \tag{7}$$

at infinity, i.e. near $\zeta = i\infty$. When $Q = 0$, (7) is in agreement with (1) and if it is assumed that the relation between the z- and ζ-planes is exactly as indicated in figure 1, the approximation of replacing material lines by free streamlines is equivalent to neglecting the effect of Q on the form (1) takes on $\eta = 0$, i.e. on

$$\phi = 2a\{\cos\alpha - \cos(\gamma+\alpha) + (\gamma+2\alpha)\sin\alpha\}. \tag{8}$$

For example (1) and (7) agree if $-iQ\zeta/2\pi$ is added to (1), and this choice does not affect (8) at all. However, in general ϕ will depend rather weakly on Q, especially if $Q/UL \ll 1$, and the theory to be given below neglects this dependence. It should be noted that the approximation introduced here is in the *mapping* relating the planes of the *independent* variables (w and ζ); our dependent variable is τ, which is related to the *derivative* of w with respect to z, and therefore the dependence of τ on Q found later in the paper is not—at least to first order in Q—inconsistent with the neglect of Q in the (w,ζ)-relationship. Further comments on this important approximation will be given in §5.

3. The general solution

The solution to the rather complicated boundary-value problem defined by (i), (ii), (vi) and (vii) is readily found as a special case of a solution given by the author (Woods 1961, pp. 148–52) for an analytic function in a rectangle, satisfying Riemann–Hilbert (mixed) conditions on a pair of opposite sides and semi-periodic conditions like (vi) and (vii) on the other pair of sides. With the parameter ϵ used in the author's text taking the values $\epsilon = \frac{1}{2}$ in $-\pi < \gamma < \mu_2$, $\epsilon = 0$ in $\mu_2 < \gamma < \mu_1$ and $\epsilon = -\frac{1}{2}$ in $\mu_1 < \gamma < \pi$, and with the rectangle height increased to infinity, the solution just mentioned reduces to

$$\tau(\zeta) = \frac{F(\zeta)}{2\pi\cos\frac{1}{2}\zeta}\left\{\int_{\mu_2}^{\mu_1}\theta_s(\gamma)\frac{\cot\frac{1}{2}(\gamma-\zeta)}{\sec\frac{1}{2}\gamma F(\gamma)}d\gamma + \int_f \Omega_s(\gamma)E(\gamma)\cos\tfrac{1}{2}\gamma\cot\tfrac{1}{2}(\gamma-\zeta)\,d\gamma\right.$$
$$\left. + \int_0^\infty \frac{\omega(\eta)\cosh\frac{1}{2}\eta}{\cosh\eta+\cos\zeta}\left[\sin\zeta\,\mathbf{Re}\left(\frac{1}{F(i\eta)}\right)+\sinh\eta\,\mathbf{Im}\left(\frac{1}{F(i\eta)}\right)\right]d\eta\right\}, \tag{9}$$

where

$$F(\zeta) \equiv [\sin\tfrac{1}{2}(\mu_1-\zeta)\sin\tfrac{1}{2}(\zeta-\mu_2)]^{\frac{1}{2}}, \quad E(\gamma) \equiv |[\sin\tfrac{1}{2}(\gamma-\mu_1)\sin\tfrac{1}{2}(\gamma-\mu_2)]^{-\frac{1}{2}}|,$$

and

$$\int_f \equiv \left(\int_{\mu_1}^{\pi} - \int_{-\pi}^{\mu_2}\right).$$

The awkward functions appearing in the last integral of (9) make it very difficult to develop further the theory for the general case. With symmetrical bodies at zero mean incidence, $\mathbf{Im}\,F(i\eta) = 0$, and so the mathematical difficulties are reduced and some progress can be made with a vortex sheet present; this case will be discussed in a later paper. For the present we shall consider only those unsteady motions for which the effect of any vortex sheet, lying to the rear of the cavity, on the flow in the neighbourhood of the hydrofoil can be neglected. In this case the last integral in (9) can be omitted.

Because of the factor $\sec\frac{1}{2}\zeta$ in (9), condition (iv) will be satisfied only if

$$\int_{\mu_2}^{\mu_1}\theta_s(\gamma)\frac{\sin\frac{1}{2}\gamma}{F(\gamma)}d\gamma + \int_f \Omega_s(\gamma)E(\gamma)\sin\tfrac{1}{2}\gamma\,d\gamma = 0,$$

a result that enables us to write (9) in the form (neglecting the vortex sheet term)

$$\tau(\zeta) = \frac{F(\zeta)}{2\pi}\left\{\int_{\mu_2}^{\mu_1} \theta_s(\gamma)\frac{\operatorname{cosec}\frac{1}{2}(\gamma-\zeta)}{F(\gamma)}\,d\gamma + \int_f \Omega_s(\gamma)\,E(\gamma)\operatorname{cosec}\tfrac{1}{2}(\gamma-\zeta)\,d\gamma\right\}. \quad (10)$$

Expanding (10) in a power series in $e^{i\zeta}$, and noting from (ii) that the first term must be $O(e^{i\zeta})$, we find that

$$\int_{\mu_2}^{\mu_1} \theta_s(\gamma)\frac{e^{-\frac{1}{2}i\gamma}}{F(\gamma)}\,d\gamma + \int_f \Omega_s(\gamma)\,e^{-\frac{1}{2}i\gamma}\,E(\gamma)\,d\gamma = 0, \quad (11)$$

of which the imaginary part is the result already found from (iv), and

$$\tau(\zeta) = i e^{i\beta} Y e^{i\zeta} + O(e^{2i\zeta}), \quad (12)$$

where $$Y \equiv \frac{1}{2\pi}\int_{\mu_2}^{\mu_1} \theta_s(\gamma)\frac{e^{-\frac{3}{2}i\gamma}}{F(\gamma)}\,d\gamma + \frac{1}{2\pi}\int_f \Omega_s(\gamma)\,E(\gamma)\,e^{-\frac{3}{2}i\gamma}\,d\gamma, \quad (13)$$

and $\beta \equiv \frac{1}{4}(\mu_1+\mu_2)$. Now (iv) can be written

$$\int_{-\pi+i\eta}^{\pi+i\eta} e^{\tau(\zeta)}\,(dw/d\zeta)\,d\zeta = 0, \quad (14)$$

and on using (2), (7) and (12) in this integrand we find that

$$2\sin\alpha - iQ/(2\pi a) = e^{i(\beta-\alpha)}Y. \quad (15)$$

We must now calculate $\Omega_s(\gamma)$. As our choice of origin for ϕ makes $\partial\phi/\partial t$ zero at the front stagnation point, Bernoulli's equation can be written

$$\left(\frac{q}{U}\right)^2 = 1+\sigma-\frac{2}{U^2}\frac{\partial\phi}{\partial t}, \quad (16)$$

where $$\sigma \equiv (p_0-p_c)/(\tfrac{1}{2}\rho U^2)-1, \quad (17)$$

is the (time-dependent) cavitation number defined in terms of the stagnation pressure p_0 and the cavity pressure p_c. We choose the stagnation pressure in defining σ, rather than the pressure at infinity, because as remarked in §1, this latter pressure is infinite unless Q is zero.

If the theory is restricted to cases for which

$$(2/U^2)(\partial\phi/\partial t) \ll 1+\sigma \quad (18)$$

(cf. remark in §5) it follows from (4), (8) and (16) that

$$\Omega_s(\gamma) = A - B\cos\gamma + C(\gamma+\sin\gamma)+\Omega'(\gamma), \quad (19)$$

where $$A \equiv -\tfrac{1}{2}\ln(1+\sigma)+b(\cos\alpha+2\alpha\sin\alpha), \quad B \equiv b\cos\alpha,$$

$$C \equiv b\sin\alpha, \quad b = \frac{a}{U^4(1+\sigma)^2}\frac{\partial}{\partial t}\{U^2(1+\sigma)\},$$

and $\Omega'(\gamma)$ is zero except in the small intervals

$$-\epsilon_1+\pi < \gamma < \pi, \quad -\pi < \gamma < -\pi+\epsilon_2,$$

i.e. in the region $CBB'C'$ at the rear of the cavity. The integral involving $\Omega'(\gamma)$ will be simplified by assuming ϵ_1 and ϵ_2 to be small, so that

$$\frac{1}{2\pi}\int_f \Omega'(\gamma)\,E(\gamma)\operatorname{cosec}\tfrac{1}{2}(\gamma-\zeta)\,d\gamma \simeq S\sec\tfrac{1}{2}\zeta,$$

where $$S \equiv \frac{1}{2\pi}(\cos\tfrac{1}{2}\mu_1\cos\tfrac{1}{2}\mu_2)^{-\frac{1}{2}}\left(\int_{-\epsilon_1+\pi}^{\pi}+\int_{-\pi}^{\epsilon_2-\pi}\right)\Omega'(\gamma)\,d\gamma.$$

This is equivalent to using a singularity to close the cavity. Incidentally an obvious physical restriction on $\Omega'(\gamma)$ is that the singularity should provide no lift force to the combined hydrofoil-cavity 'body'.

4. The principal equations

The following notation will be adopted:

$$c \equiv \cos\mu,\quad s = \sin\mu,\quad \mu \equiv \tfrac{1}{2}(\mu_1-\mu_2),\quad h \equiv \tfrac{1}{2}e^{-i\beta}(1+ce^{-2i\beta}),$$
$$b \equiv \tfrac{1}{2}e^{-3i\beta}\{c+\tfrac{1}{2}e^{-2i\beta}(3c^2-1)\},$$
$$g \equiv e^{-i\beta}\{2\ln\sin\tfrac{1}{2}\mu - i(\pi-\mu)\},\quad \lambda \equiv i\{\sin\tfrac{1}{2}(\zeta-\mu_2)\operatorname{cosec}\tfrac{1}{2}(\mu_1-\zeta)\}^{\frac{1}{2}},$$
$$I \equiv \cos\zeta - F(\zeta)\sec\tfrac{1}{2}\zeta\{\cos(\zeta+\beta)+\cos\beta\},$$
$$K \equiv \sin\zeta - F(\zeta)\sec\tfrac{1}{2}\zeta\{\sin(\zeta+\beta)+\sin\beta\} + 2i\ln\{(\lambda-e^{i\frac{1}{2}\mu})/(\lambda+e^{-i\frac{1}{2}\mu})\}.$$

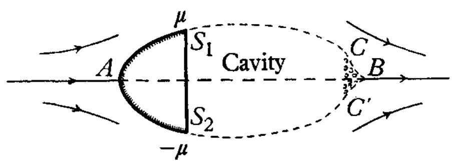

FIGURE 2. Symmetrical cavity.

Then when (19) is substituted into (10) the result can be written

$$\tau(\zeta) = \frac{F(\zeta)}{2\pi}\int_{\mu_2}^{\mu_1}\theta_s(\gamma)\frac{\operatorname{cosec}\tfrac{1}{2}(\gamma-\zeta)}{F(\gamma)}\,d\gamma + A - BI + CK + SF(\zeta)\sec\tfrac{1}{2}\zeta, \quad (20)$$

which is not, of course, valid near the singularity at $\zeta = \pm\pi$.

On expanding (20) near infinity we find that

$$\frac{i}{2\pi}\int_{\mu_2}^{\mu_1}\theta_s(\gamma)\frac{e^{-i\frac{1}{2}\gamma}}{F(\gamma)}\,d\gamma + Ae^{-i\beta} - B(h-\cos\beta) + iC(h+g+i\sin\beta) + S = 0, \quad (21)$$

and $$Y = \frac{1}{\pi}\int_{\mu_2}^{\mu_1}\theta_s(\gamma)\cos\tfrac{1}{2}\gamma\frac{e^{-i\gamma}}{F(\gamma)}\,d\gamma - 2ihA + i(b-i\sin\beta)B$$
$$+ C[b+i\sin\beta+(1+c)e^{-3i\beta}+2hge^{i\beta}] = e^{i(\alpha-\beta)}\left(2\sin\alpha - \frac{iQ}{2\pi a}\right), \quad (22)$$

corresponding to (11) and (15).

With a symmetrical body and cavity as shown in figure 2, these equations are much simplified. In this case $\theta_s(\gamma)$ is an odd function of γ, while $\Omega_s(\gamma)$ is an even function. Also β, α and C are all zero. Equation (20) becomes

$$\tau(\zeta) = \frac{2F(\zeta)}{\pi}\cos\tfrac{1}{2}\zeta\int_0^{\mu}\frac{\theta_s(\gamma)\sin\tfrac{1}{2}\gamma\,d\gamma}{F(\gamma)(\cos\zeta-\cos\gamma)} + A - BI(\zeta) + SF(\zeta)\sec\tfrac{1}{2}\zeta, \quad (23)$$

where $I(\zeta) = \cos\zeta - 2\cos\frac{1}{2}\eta F(\zeta)$ and $F(\zeta) = \{\frac{1}{2}(\cos\zeta - c)\}^{\frac{1}{2}}$. Also for this case

$$A = -\tfrac{1}{2}\ln(1+\sigma) + B, \quad B = b.$$

Equations (21) and (22) reduce to

$$\frac{1}{\pi}\int_0^{\mu} \theta_s(\gamma)\frac{\sin\frac{1}{2}\gamma}{F(\gamma)}\,d\gamma + A + \tfrac{1}{2}(1-c)\,B + S = 0, \tag{24}$$

and $$iY = \frac{2}{\pi}\int_0^{\mu} \theta_s(\gamma)\cos\tfrac{1}{2}\gamma\frac{\sin\gamma}{F(\gamma)}\,d\gamma + (1+c)\,A - \tfrac{1}{4}(1+c)\,(3c-1)\,B = \frac{Q}{2\pi a}. \tag{25}$$

The integral over the wetted surface appearing in (20), (21) and (22) can be evaluated with the assistance of the indefinite integrals

$$\frac{F(\zeta)}{2\pi}\int \frac{\operatorname{cosec}\frac{1}{2}(\gamma-\zeta)}{F(\gamma)}\,d\gamma = -\frac{2}{\pi}\tanh^{-1}\left\{\frac{\sin\frac{1}{2}(\mu_1-\gamma)\sin\frac{1}{2}(\zeta-\mu_2)}{\sin\frac{1}{2}(\gamma-\mu_2)\sin\frac{1}{2}(\mu_1-\zeta)}\right\}^{\frac{1}{2}} \equiv J(\gamma,\zeta), \tag{26}$$

$$\frac{1}{2\pi}\int \frac{i\,e^{-i\frac{1}{2}\gamma}}{F(\gamma)}\,d\gamma = e^{-i\beta}J(\gamma, i\infty),$$

and $$\frac{1}{2\pi}\int \frac{\cos\frac{1}{2}\gamma\, e^{-i\gamma}}{F(\gamma)}\,d\gamma = \frac{2}{\pi}\frac{F(\gamma)\sin\frac{1}{2}\mu\cos\frac{1}{2}\gamma}{\sin(\frac{1}{2}\gamma-\beta)}e^{-3i\beta} - 2ihJ(\gamma, i\infty).$$

Notice from (26) that the integral in (20) can be written

$$\int_{\mu_2}^{\mu_1} \theta_s(\gamma)\,dJ = |\theta_s(\gamma)\,J(\gamma,\zeta)|_{\mu_2}^{\mu_1} - \int_{\mu_2}^{\mu_1} J(\gamma,\zeta)\frac{d\phi/d\gamma}{Rq}\,d\gamma, \tag{27}$$

where R is the radius of curvature of the wetted surface. As $\mathbf{Re}\,\tau = \ln(U/q)$, this form enables us to change the real part of (20) into a rather involved integral equation for q, which must, in general, be solved by an iterative procedure (see Woods 1961, pp. 449–54). When $\theta_s(\gamma)$ is a step function, i.e. when the body is a polygon, this difficulty does not arise, for the integral on the right-hand side of (27) vanishes.

5. Examples

(i) *A symmetrical wedge*

The simplest example of the theory, but still one of considerable interest, is the symmetrical wedge (see figure 3). In this case $\theta_s(\gamma) = -\theta_s(-\gamma) = \alpha_0$, and (23), (24) and (26) yield the general solution

$$\tau(\zeta) = -2\alpha_0\left\{\tfrac{1}{2}i + J(0,\zeta) - \frac{F(\zeta)}{2\pi\cos\frac{1}{2}\zeta}\ln\left(\frac{1-m}{1+m}\right)\right\} + A\{1 - F(\zeta)\sec\tfrac{1}{2}\zeta\}$$
$$- B\{I(\zeta) + \tfrac{1}{2}(1-c)\,F(\zeta)\sec\tfrac{1}{2}\zeta\}, \tag{28}$$

where $J(0,\zeta) = -(2/\pi)\tanh^{-1}\{\sin\frac{1}{2}(\zeta+\mu)\operatorname{cosec}\frac{1}{2}(\mu-\zeta)\}^{\frac{1}{2}}$, and $m \equiv \sin\frac{1}{2}\mu$. On expanding (28) near infinity, one finds, corresponding to (25), that

$$\frac{Q}{2\pi a} = \frac{4\alpha_0}{\pi}\left\{m - \tfrac{1}{2}(1-m^2)\ln\left(\frac{1-m}{1+m}\right)\right\} - (1-m^2)\ln(1+\sigma) + (1-m^2)\,(1+3m^2)\,b. \tag{29}$$

These equations are exact; but we shall now apply linear perturbation theory to find some approximate results.

From (1) $\phi = 2a(1-\cos\gamma)$, so that if l is the length of the wedge and the angle α_0 is small, $l \simeq (4a/U)m^2$, and by (3)

$$a \simeq \tfrac{1}{4}UL, \quad \text{so} \quad m \simeq (l/L)^{\frac{1}{2}}. \tag{30}$$

For cavities long compared with the length of the wedge it follows from the above equations that

$$\frac{2Q}{\pi LU} = \frac{8\alpha_0}{\pi}\left(\frac{l}{L}\right)^{\frac{1}{2}} - \sigma + \frac{L}{2U^2}(\dot{U} + \tfrac{1}{2}\dot{\sigma}U). \tag{31}$$

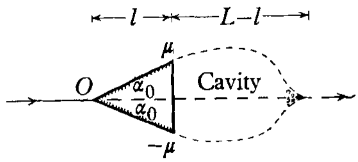

FIGURE 3. Symmetrical wedge.

If $\dot{U} = 0$, (18) will be satisfied provided $L\dot{\sigma} \ll U$, i.e. provided that the rear of the cavity moves with a velocity much less than that of the liquid itself. And the restriction on Q/UL mentioned in §2, viz. $Q/UL \ll 1$, is a consequence of (18) and (31).

The drag coefficient $C_D \equiv D/(\frac{1}{2}l\rho U^2)$, where D is the drag per unit length of the wedge can be readily calculated for a wedge of small apex angle $2\alpha_0$. Linear perturbation theory in this case yields

$$C_D = \frac{2\alpha_0}{Ul}\int_0^{\mu}\{(p-p_c)/(\tfrac{1}{2}\rho U^2)\}\,dx(\gamma) = \frac{2\alpha_0 L}{l}\int_0^{\mu}\{\sigma + 2\Omega - 2B(1-\cos\gamma)\}\sin\gamma\,d\gamma,$$

by (3) and $x \simeq UL(1-\cos\gamma)$. Here Ω is the real part of (28). On carrying out the integration and then using (29) to eliminate $\ln(1+\sigma) \simeq \sigma$, we arrive at

$$C_D = \frac{8\alpha_0^2}{\pi}\frac{1}{1-l/L} + 2\alpha_0\left\{\frac{m}{U^2}L(\dot{U}+\tfrac{1}{2}\dot{\sigma}U)\frac{2Q}{\pi Ul}\left[\frac{m}{1-m^2} - \tfrac{1}{2}\ln\left(\frac{1+m}{1-m}\right)\right]\right\}. \tag{32}$$

For the special case of constant volume cavities, $Q = 0$, and we find that (29), (30) and (32) give results agreeing with Cumberbatch's (1961) theory, based on the acceleration potential, which for linearized theory is defined by

$$(p_c - p)/(\rho q_c^2).$$

The steady forms of (29) and (32), i.e. the forms in which both Q and $\partial/\partial t$ are zero, agree with formulae given by Wu (1957).

For constant-volume cavities (see remarks in Introduction), (31) and (32) give

$$C_D = \frac{8\alpha_0^2}{\pi}\left(1 - \frac{4l}{L}\right) + 4\alpha_0\left(\frac{l}{L}\right)^{\frac{1}{2}}\sigma. \tag{33}$$

(ii) *Flat plate hydrofoil*

This case is shown in figure 4. The incidence is α_0, and consequently

$$\theta_s(\gamma) = -\alpha_0 + \pi\mathbf{U}(\gamma + 2\alpha),$$

where $\mathbf{U}(x)$ is the unit function. Then from (20), (21) and (26)

$$\tau(\zeta) = -\{i\alpha_0 + \pi J(-2\alpha, \zeta)\} + F(\zeta)\sec\tfrac{1}{2}\zeta\{\alpha_0 \sin\beta + \pi\,\mathbf{Re}\,[e^{-i\beta}J(-2\alpha, i\infty)]\} - F(\zeta)\sec\tfrac{1}{2}\zeta\,(AM(\zeta) + BN(\zeta) - CT(\zeta)), \quad (34)$$

where

$$M(\zeta) \equiv 1 - F(\zeta)\sec\tfrac{1}{2}\zeta\cos\beta, \quad N(\zeta) \equiv I(\zeta) + (\cos\beta - \mathbf{Re}\,h)\,F(\zeta)\sec\tfrac{1}{2}\zeta,$$

and

$$T(\zeta) \equiv K(\zeta) + \{\sin\beta + \mathbf{Im}(h-g)\}\,F(\zeta)\sec\tfrac{1}{2}\zeta.$$

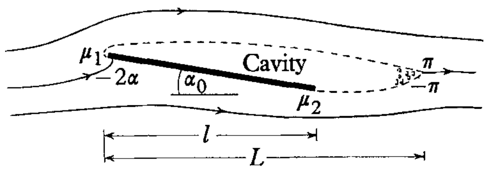

FIGURE 4. Flat-plate hydrofoil.

If α_0 is small, then α will also be small. Then $\mu_1 \simeq -2\alpha \simeq 0$, $\mu_2 = -2\mu$ and $\beta \simeq -\frac{1}{2}\mu$. With the notation $\delta^2 \equiv (\alpha + \frac{1}{2}\mu_1)$, so that δ is a small number, we can write (34) in the form

$$\tau(\zeta) = -i\{\alpha_0 + 2\delta[\sin(\tfrac{1}{2}\zeta + \mu)\operatorname{cosec}\tfrac{1}{2}\zeta]^{\frac{1}{2}}\} - F(\zeta)\sec\tfrac{1}{2}\zeta[\alpha_0\sin\tfrac{1}{2}\mu + AM(\zeta) + BN(\zeta) - CT(\zeta)]. \quad (35)$$

Expanding this near infinity we find that

$$\alpha_0\cos\tfrac{1}{2}\mu + \frac{2\delta}{(\sin\mu)^{\frac{1}{2}}} + A\sin\beta + B\,\mathbf{Im}\,h - C\,\mathbf{Re}\,(h+g) = 0, \quad (36)$$

which serves to determine the value of δ. From (12), (15), (35) and (36) we find the relations

$$\frac{Q}{2\pi a(1+c)} = \alpha_0 s + Ac - \tfrac{1}{4}B(3c-1)\cos 2\mu, \quad (37)$$

$$\frac{2\alpha}{1-c} = \alpha_0 c - As + \tfrac{1}{4}B(3c+1)\sin 2\mu. \quad (38)$$

Equation (37) plays the same role for the hydrofoil as (29) does for the wedge, while (38), together with (2), provide a formula for the circulation about the hydrofoil-cavity 'body'.

Finally note that for this case the equation corresponding to (30) is

$$\sin^2\mu \simeq l/L,$$

and that the lift and drag on the hydrofoil can be calculated from

$$C_L + iC_D = (1 + i\alpha_0)\frac{L}{2l}\int_{-2\mu}^{0}\{\sigma + 2\Omega - 2B(1-\cos\gamma)\}\sin\gamma\, d\gamma$$

where Ω is the real part of (35).

6. Extension of theory to a cascade of hydrofoils

The extension of the foregoing theory to the cascade of hydrofoils shown in the z-plane of figure 5 is quite straightforward. Let α be the stagger angle, and H the gap distance; then HU is the interval in the w-plane between corresponding

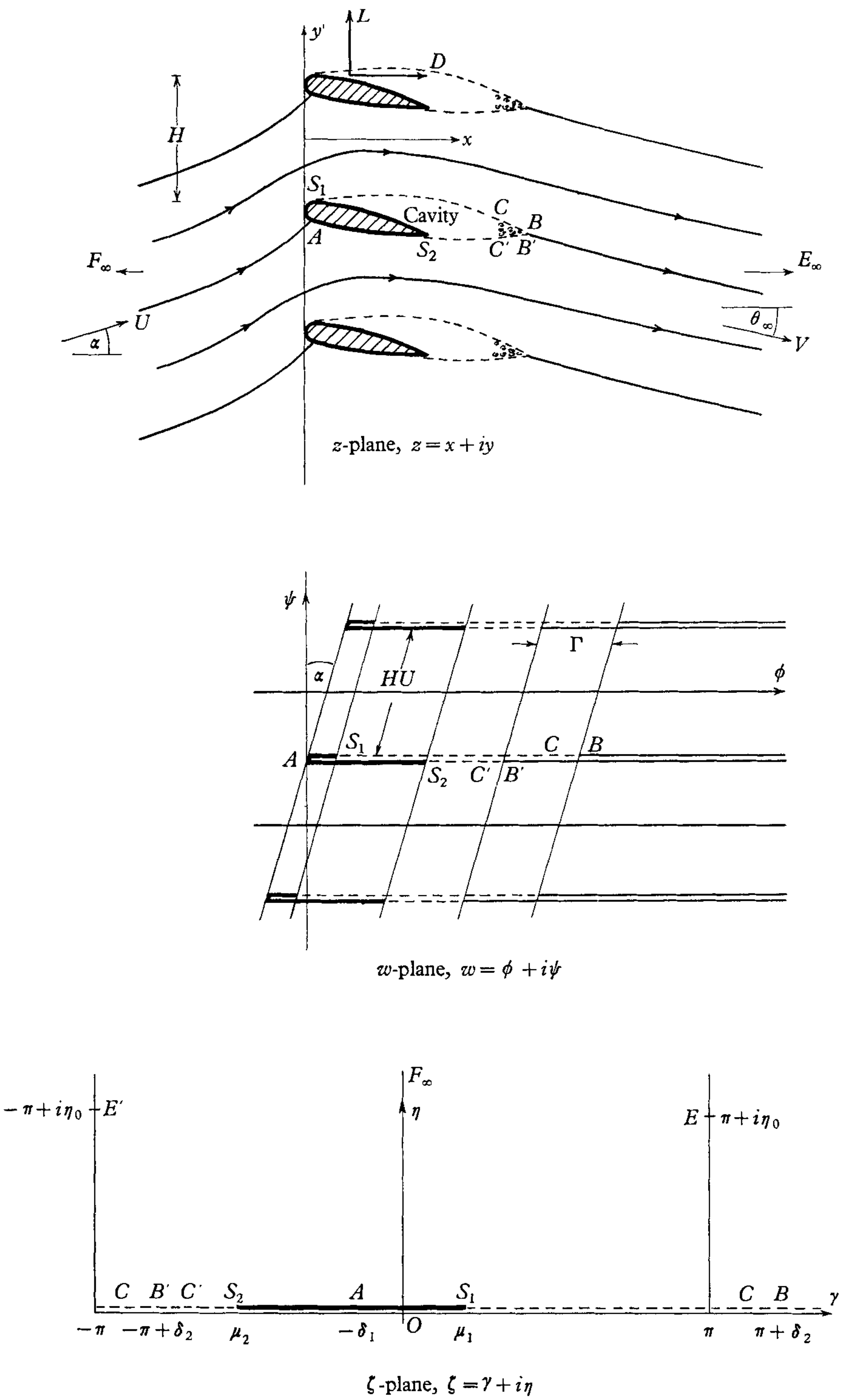

FIGURE 5. The conformal transformations for a cascade of hydrofoils.

points on adjacent hydrofoils. The relation between the w-plane and the ζ-plane can be shown to be (see Woods 1961, pp. 490–3)

$$w = \frac{HU}{2\pi}\zeta\sin\alpha + \frac{1}{\pi}(\Gamma - HU\sin\alpha)\tan^{-1}(e^{-r}\tan\tfrac{1}{2}\zeta) - \frac{HU}{2\pi}\cos\alpha\ln\left(\frac{1+T\cos\zeta}{1+T}\right), \tag{39}$$

where $r \equiv (4a/HU)$, $T \equiv \tanh r$ and a is a parameter approximately equal to $\frac{1}{4}LU$, corresponding to the number a introduced in (1). The point E_∞ down stream at infinity is mapped on to $\gamma = \pi$, $\eta = \eta_0$, where

$$\operatorname{sech}\eta_0 = \tanh r = T, \tag{40}$$

while the point upstream at infinity is at $\eta = \infty$ as before. The front stagnation point A is mapped on to $\gamma = -\delta_1$ and the rear (cusped) end B is mapped on to $\gamma = \pm\pi + \delta_2$, where δ_1 and δ_2 are given by the relations $\delta_1 = \delta + \alpha$, $\delta_2 = \delta - \alpha$, with $\delta \equiv \sin^{-1}\{(\sin\alpha\cosh r - \cos\alpha\tan\theta_\infty)/\sinh r\}$.

The general solutions contained in (9) and (10) are still applicable, but (11) is now replaced by

$$\int_{\mu_2}^{\mu_1}\theta_s(\gamma)\frac{\cos\frac{1}{2}\gamma\,d\gamma}{F(\gamma)} + \int_f \Omega_s(\gamma)\cos\tfrac{1}{2}\gamma\,E(\gamma)\,d\gamma = \alpha, \tag{41}$$

as
$$\lim_{\eta\to\infty}\tau = i\alpha.$$

In place of (15) we have

$$\lim_{\zeta\to\pi+i\eta_0}\tau = \tau_\infty = \Omega_\infty + i\theta_\infty,$$

where τ_∞ is the value of τ downstream at infinity. Thus from (10)

$$\Omega_\infty + i\theta_\infty = -\frac{F(\pi+i\eta_0)}{2\pi}\left\{\int_{\mu_2}^{\mu_1}\theta_s(\gamma)\frac{\sec\frac{1}{2}(\gamma-i\eta_0)}{F(\gamma)}d\gamma + \int_f \Omega_s(\gamma)\,E(\gamma)\sec\tfrac{1}{2}(\gamma-i\eta_0)\,d\gamma\right\}, \tag{42}$$

an equation that ensures the closure of the hydrofoil-cavity body. In place of (19) there is a similar equation based on the form (39) takes on $\eta = 0$. The general case is rather complicated; when Γ and α are zero the equation in question is

$$\Omega_s(\gamma) = A - B\ln(1+T\cos\gamma), \tag{43}$$

where
$$A \equiv -\tfrac{1}{2}\ln(1+\sigma) + B\ln(1+T), \quad B \equiv \frac{HUb}{4\pi a}.$$

On substituting (43) into (9) and ignoring the vortex sheet term, we find

$$\tau(\zeta) = \frac{F(\zeta)}{2\pi\cos\frac{1}{2}\zeta}\int_{\mu_2}^{\mu_1}\theta_s(\gamma)\frac{\cot\frac{1}{2}(\gamma-\zeta)}{F(\gamma)\sec\frac{1}{2}\gamma}d\gamma + AM(\zeta) - BG(\zeta), \tag{44}$$

where

$$\frac{\partial G}{\partial T} \equiv \frac{M(\zeta)}{T} - \frac{1}{T(1-T)}\left\{\frac{1-(\bar{\nu}/\nu)F(\zeta)\sec\frac{1}{2}\zeta\,[e^{-r}\cos\bar{\beta} + \frac{1}{2}\sin\zeta\sin\bar{\beta}(1-e^{-2r})]}{\sin^2\frac{1}{2}\zeta + e^{2r}\cos^2\frac{1}{2}\zeta}\right\},$$

and
$$\nu \equiv (\cos\tfrac{1}{2}\mu_1\cos\tfrac{1}{2}\mu_2), \quad \bar{\beta} \equiv \tfrac{1}{4}(\bar{\mu}_1+\bar{\mu}_2), \quad \bar{\mu}_1 \equiv 2\tan^{-1}(e^{-r}\tan\tfrac{1}{2}\mu_1),$$
$$\bar{\mu}_2 \equiv 2\tan^{-1}(e^{-r}\tan\tfrac{1}{2}\mu_2), \quad \bar{\nu} \equiv (\cos\tfrac{1}{2}\bar{\mu}_1\cos\tfrac{1}{2}\bar{\mu}_2)^{\frac{1}{2}}.$$

For a cascade of wedges the solution is now given by replacing the integral in (44) by the first bracketed term on the right-hand side of (28); similarly, for a cascade of hydrofoils one uses the first bracketed term in (34). When linear perturbation theory is applicable, the lift and drag on a member of the cascade can be obtained from an equation like the last in §5, except that $UL \sin\gamma \, d\gamma$ must be replaced by $d\phi$, where $d\phi$ is the derivative of (39) on the wetted surface, $\eta = 0$.

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